

RELATIVE $\bar{\partial}$ -COMPLEX AND ITS CURVATURE PROPERTIES

XU WANG

ABSTRACT. We shall define the relative $\bar{\partial}$ -complex and study the curvature properties of the associated vector bundles. As an application, we shall prove that Yamaguchi's theory on subharmonicity of the Green operator can be seen as a curvature property of the quotient bundle. A short survey of other recent applications will also be given in this paper.

MATHEMATICS SUBJECT CLASSIFICATION (2010): 32A25

KEYWORDS: $\bar{\partial}$ -operator, subbundle, quotient bundle, Hörmander L^2 -theory, Yamaguchi's theory, Berndtsson's complex Brunn-Minkowski theory.

1. INTRODUCTION

Our motivation to write this paper is to give a unified proof of recent results in [13], [32], [33] and [34] by using the relative $\bar{\partial}$ -complex. Let π be a **holomorphic submersion** from a complex manifold \mathcal{X} to a complex manifold B , let E be a holomorphic vector bundle over \mathcal{X} , let X_t be the fibre at t and E_t be the restriction of E to X_t . Denote by $\mathcal{A}_t^{p,q}$ the space of smooth E_t -valued (p, q) -forms on X_t . We call the collection of $\bar{\partial}$ -operators on fibres

$$\{\bar{\partial}^t\}_{t \in B} : \{\mathcal{A}_t^{p,q}\}_{t \in B} \rightarrow \{\mathcal{A}_t^{p,q+1}\}_{t \in B},$$

the **relative $\bar{\partial}$ -complex**. Put $\mathcal{A}^{p,q} = \{\mathcal{A}_t^{p,q}\}_{t \in B}$, $\mathcal{K}^{p,q} := \{\text{Ker } \bar{\partial}^t\}_{t \in B}$, $\mathcal{I}^{p,q+1} := \{\text{Im } \bar{\partial}^t\}_{t \in B}$. Then we have the following exact sequence

$$(1.1) \quad 0 \rightarrow \mathcal{K}^{p,q} \rightarrow \mathcal{A}^{p,q} \rightarrow \mathcal{I}^{p,q+1} \rightarrow 0.$$

We call $\mathcal{K}^{p,q}$ the **$\bar{\partial}$ subbundle** and $\mathcal{I}^{p,q+1}$ the **$\bar{\partial}$ quotient bundle**. In general, they are not holomorphic vector bundles but still we can define the Chern connection on them as an **operator on the space of smooth sections** (see [2] and [23] for related results). Our starting point is the following result of Berndtsson (see Theorem 1.1 in [2]):

If π a product family then one may study the curvature properties of $\mathcal{K}^{n,0}$, n is the fibre dimension, by looking at $\mathcal{K}^{n,0}$ as a holomorphic subbundle of $\mathcal{A}^{n,0}$.

The curvature properties of $\mathcal{K}^{n,0}$ are crucial in Berndtsson's **complex Brunn-Minkowski theory**, which also contains curvature properties of vector bundles associated to a non-trivial fibration (see [1], [2], [5], [3], [7], [9], [11], [10], [12], see also [31], [30], [24], [16], [27], [28] and [32] for other generalizations). Thus it is natural to ask: **Whether the relative $\bar{\partial}$ -complex can be used to prove the curvature properties of the associated vector bundles for a general fibration?** We shall try to answer this question in this paper. In particular, we shall show that the relative $\bar{\partial}$ -complex can be used to give a unified proof of recent results in [13], [32], [33] and [34].

Date: July 14, 2016.

Research supported by Knut and Alice Wallenberg Foundation, and the China Postdoctoral Science Foundation.

Another related theory is **Yamaguchi's theory** on subharmonicity properties of the Green-operators (see [35], [25], [26], [22] and [33]). In the KAWA-NORDAN 2014 conference in Marseille, Levenberg asked the following question:

Is Yamaguchi's theory a curvature property ?

In this paper, we shall answer Levenberg's question by proving that **Yamaguchi's theory can be seen as a curvature property of the quotient bundle $\mathcal{I}^{n,n}$** .

2. FAMILY OF DOMAINS IN A FIXED MANIFOLD

First we shall consider variation of domains X_t , $t \in B$, in a **fixed** complex manifold M . We shall show how to use the **curvature properties of the $\bar{\partial}$ -quotient bundle** to study **variation of the L^2 -minimal solution of the $\bar{\partial}$ -operator on fibres**, see [33].

Let (E_0, h^0) be a **fixed** holomorphic vector bundle over (M, ω^0) , let g be a **fixed** smooth $\bar{\partial}$ -closed E_0 -valued $(n, q+1)$ -form on M . We shall introduce the following definition:

Definition 2.1. *If a^t is the L^2 -minimal solution of $\bar{\partial}^t(a^t) = g$ on X_t with respect to ω^0 and h^0 then we call $\|a^t\|$ the **Green norm** of g on X_t and denote it by $\|g\|_G(t)$.*

Remark 1: If the above $\bar{\partial}$ -equation has no L^2 -solution then we say the Green norm of g is infinite on X_t .

Relation with Green operator: Since the L^2 -minimal solution a^t of $\bar{\partial}^t(a^t) = g$ can be written as $a^t = (\bar{\partial}^t)^* G^t g$, where each G^t is the **Green-operator** with respect to the $\bar{\partial}^t$ -Laplacian on X_t . Thus we have

$$\|g\|_G^2(t) = \|a^t\|^2 = ((\bar{\partial}^t)^* G^t g, a^t) = (G^t g, g).$$

That is why we call the L^2 -norm of the minimal solution of the $\bar{\partial}$ -equation the Green-norm. See [1] for the **relation between the Green-norm and the Robin constant**.

Relation with the quotient norm: Since the minimal solution is just the **minimal lift** w.r.t. the **relative $\bar{\partial}$ -complex**: $\bar{\partial}^t : \mathcal{A}_t^{p,q} \rightarrow \mathcal{I}_t^{p,q+1}$, we know that **the Green norm is just the quotient norm**.

Now we know that one may use curvature formula of the quotient bundle to study variation of the Green norm. The latter is known as **Yamaguchi's theory**, thus **Yamaguchi's theory can be seen as a curvature property of the quotient bundle**, this answers the previous question raised by Levenberg. Now we have another formulation of Yamaguchi's theory:

Theorem 2.1 (Developed by Yamaguchi-Maitani-Levenberg-Kim, etc). *Under some convexity and curvature assumptions, $-\|g\|_G^2$ is plurisubharmonic if g is a holomorphic section of the quotient bundle $\mathcal{I}^{n,n}$, where n is the fibre dimension.*

Our first result is a generalization of Yamaguchi's theory to the $(n, q+1)$ -case. More precisely, we shall prove the following theorem (see the main theorem in [33]):

Theorem 2.2. *[$I^{n,q+1}$ -version of Yamaguchi's theorem] Let g be a $\bar{\partial}$ -closed E^0 -valued $(n, q+1)$ -form on M . Assume that the total space $\mathcal{X} := \{(z, t) \in M \times B : z \in X_t\}$ is Stein, ω^0 is Kähler and*

$$(2.1) \quad i\Theta(E_0, h^0) \wedge (\omega^0)^q \geq 0,$$

on M . Assume further g has **compact support in each fibre** X_t . Then $-||g||_G^2$ is plurisubharmonic on B .

Main tools in the proof: Hörmander's L^2 -theory [20] and the curvature formula for the $\bar{\partial}$ quotient bundle $\mathcal{I}^{n,q+1}$. By a generalized version of Berndtsson's approximation process (see [1] or [33] for details), it suffices to prove the following **product case** with a non-product weight ψ (a real smooth function on $M \times B$):

Theorem 2.3 (Product case with non-product weight ψ). *Assume that $\mathcal{X} = X_0 \times B$, where X_0 is a smoothly bounded strictly pseudoconvex domain in M . Assume that the pull back to the total space, say ω , of ω^0 is Kähler and*

$$i\Theta(E, h) \wedge \omega^q > 0, \quad E := E_0 \times B, \quad h := e^{-\psi}(h^0 \times B).$$

*Assume further that g has **compact support** in X_0 and ψ **does not depend on t** on $\text{Supp}(g) \times B$. Then $-||g||_G^2$ is plurisubharmonic on B .*

Proof. Since the **Green norm is just the quotient norm**, it is enough to estimate the curvature of the quotient bundle $\mathcal{I}^{n,q+1}$.

Let us assume that B is **one dimensional**. Denote by $\Theta_{t\bar{t}}$ the curvature operators on $\mathcal{A}^{n,q}$, then we have

$$\Theta_{t\bar{t}} = [D_t, \bar{\partial}_t] = \psi_{t\bar{t}}, \quad D_t := \partial/\partial t - \psi_t, \quad \bar{\partial}_t := \partial/\partial \bar{t}.$$

Since $\mathcal{I}^{n,q+1} = \mathcal{A}^{n,q}/\mathcal{K}^{n,q}$ is the quotient bundle of $\mathcal{A}^{n,q}$, we have

$$(\Theta_{t\bar{t}}^{\mathcal{I}} g, g)_G = (\psi_{t\bar{t}} a, a) + ||P(a_{\bar{t}})||^2, \quad a : t \mapsto a^t \in \mathcal{A}_t^{n,q},$$

where $\Theta_{t\bar{t}}^{\mathcal{I}}$ is the curvature operator on $\mathcal{I}^{n,q+1}$, each a^t is the L^2 -**minimal solution** of $\bar{\partial}^t(\cdot) = g$ and P denotes the **orthogonal projection** to $\mathcal{K}^{n,q}$. Notice that **Hamilton's theory** (see [18] and [19]) implies that a is a smooth section.

Since g does not depend on t and has **compact support** in each fibre, we know that there exists a **fixed** smooth L^2 -solution, say u , of $\bar{\partial}u = g$ on X_0 . Since u is fixed, we have $u_{\bar{t}} = 0$, thus u is a **holomorphic section** of $\mathcal{A}^{n,q}$. Since g is the **image of u under the $\bar{\partial}$ -quotient map**, we know that g is a **holomorphic section** of $\mathcal{I}^{n,q+1}$, thus we have

$$(||g||_G^2)_{t\bar{t}} = ||D_t a||^2 - (\Theta_{t\bar{t}}^{\mathcal{I}} g, g)_G \leq ||D_t a||^2 - (\psi_{t\bar{t}} a, a).$$

We shall use **Hörmander's L^2 -theory to control the norm** of $D_t a$. Notice that each $D_t a$ is the L^2 -**minimal solution** of

$$\bar{\partial}^t(\cdot) = \bar{\partial}^t D_t a = [\bar{\partial}^t, D_t]a + D_t g = [\bar{\partial}^t, D_t]a = -\bar{\partial}^t \psi_t \wedge a,$$

the **third** equality follows from $D_t g = g_t - \psi_t g \equiv 0$ since ψ **does not depend on t** on $\text{Supp}(g) \times B$. By **Hörmander's $\bar{\partial}$ - L^2 -estimate**, we have

$$||D_t a||^2 \leq (Q^{-1} \bar{\partial}^t \psi_t \wedge a, \bar{\partial}^t \psi_t \wedge a),$$

where

$$Q := [i\Theta(E_0, e^{-\psi^t} h^0), \Lambda_{\omega^0}].$$

Thus

$$(||g||_G^2)_{t\bar{t}} \leq (Q^{-1} \bar{\partial}^t \psi_t \wedge a, \bar{\partial}^t \psi_t \wedge a) - (\psi_{t\bar{t}} a, a)$$

By a direct computation, we know that $i\Theta(E, h) \wedge \omega^q > 0$ implies

$$(Q^{-1} \bar{\partial}^t \psi_t \wedge a, \bar{\partial}^t \psi_t \wedge a) \leq (\psi_{t\bar{t}} a, a).$$

Thus $(||g||_G^2)_{t\bar{t}} \leq 0$. The proof is complete. \square

If we consider the subbundle $\mathcal{K}^{n,q}$ and use **Hörmander's L^2 -theory to control the second fundamental form** as Berndtsson did for the $q = 0$ case (see the proof of Theorem 1.1 in [2]) then we can prove the following theorem in [33]:

Theorem 2.4. *[(n, q)-version of Berndtsson's theorem] Let v be a **fixed** smooth E_0 -valued (n, q) -form with **compact support** in each fibre. Assume that ω^0 is **Kähler**, the total space is **Stein** and $i\Theta(E_0, h^0) \wedge (\omega^0)^q \geq 0$, then $\log \|P(v)\| : t \mapsto \|P^t(v)\|$ is plurisubharmonic on B , where each $P^t(v)$ denotes the Bergman projection of v to $\ker \bar{\partial}^t = \mathcal{K}_t^{n,q}$.*

Relation with the Ohsawa-Takegoshi theorem: If $q = 0$ then by Berndtsson-Lempert-Blocki's method [11], the above theorem (with v a general current, see section 5.3 in [32] or [33]) can be used to prove Blocki-Guan-Zhou's sharp version of the Ohsawa-Takegoshi theorem (see [29], [15], [14] and [17], to cite just a few).

3. PROPER KÄHLER FIBRATION

Now let us consider the case that $\pi : \mathcal{X} \rightarrow B$ is a **proper** fibration. We shall prove that:

Theorem 3.1 (Generalized Berndtsson-Mourougane-Takayama's theorem). *Assume that ω is Kähler and $i\Theta(E, h) \wedge \omega^q \geq 0$. Assume further that $\dim H^{n,q}(E_t)$ is a **constant**. Then*

- $\mathcal{K}^{n,q}$ is **Nakano semipositive** if $q = 0$ or $c_{j\bar{k}}(\omega) \equiv 0$;
- $R^q \pi_* \mathcal{O}(\mathcal{K}_{\mathcal{X}/B} \otimes E) \simeq \mathcal{K}^{n,q}/\mathcal{I}^{n,q}$ is **Griffiths semipositive**,

where $c_{j\bar{k}}(\omega) := \langle V_j, V_k \rangle_\omega$ and each V_j denotes the horizontal lift of $\partial/\partial t^j$ with respect to ω .

Remark: In case **E is Nakano-semipositive**, Mourougane-Takayama [28] proved that $\mathcal{K}^{n,q}/\mathcal{I}^{n,q}$ is **Nakano semipositive** (with different metric, i.e. not the quotient metric).

Proof. Recall in the **product case**, the Chern connection on $\mathcal{A}^{n,q}$ is defined by $\bar{\partial}_{tj} = \partial/\partial t^j$ and $D_{tj} = \partial/\partial t^j - \psi_j$. But now we have to use the **generalized Lie derivatives** to define the Chern connection on $\mathcal{A}^{n,q}$, i.e.,

$$D_{tj} \mathbf{u} := [\partial^E, \delta_{V_j}] \mathbf{u}, \quad \bar{\partial}_{tj} \mathbf{u} = [\bar{\partial}, \delta_{\bar{V}_j}] \mathbf{u}, \quad \Theta_{j\bar{k}} := [D_{tj}, \bar{\partial}_{t\bar{k}}],$$

where $d^E := \bar{\partial} + \partial^E$ denotes the Chern connection on E , each V_j is the **horizontal lift** of $\partial/\partial t^j$ with respect to ω and \mathbf{u} is the **representative** of a smooth section $u : t \mapsto u^t$ of $\mathcal{A}^{n,q}$, i.e. $\mathbf{u}|_{X_t} = u^t$. Since $\mathcal{K}^{n,q}$ is the **subbundle** of $\mathcal{A}^{n,q}$, we have

$$(3.1) \quad (\Theta_{j\bar{k}}^{\mathcal{K}} u, v) = (\Theta_{j\bar{k}} u, v) - (P^\perp D_{tj} u, P^\perp D_{t\bar{k}} v).$$

By Theorem 2.5 in [13], we have

$$(3.2) \quad (\Theta_{j\bar{k}} u, v) = ([L_j, L_{\bar{k}}] u, v) - (\partial \bar{V}_k|_{X_t \lrcorner} u, \partial \bar{V}_j|_{X_t \lrcorner} v) + (\bar{\partial} V_j|_{X_t \lrcorner} u, \bar{\partial} V_k|_{X_t \lrcorner} v).$$

By Proposition 4.2 in [32], we know that

$$(3.3) \quad [L_j, L_{\bar{k}}] = [d^E, \delta_{[V_j, \bar{V}_k]}] + \Theta(E, h)(V_j, \bar{V}_k).$$

Moreover, by Lemma 6.1 in [32], we know that

$$(3.4) \quad \delta_{[V_j, \bar{V}_k]} = (\partial^t c_{k\bar{j}}(\omega))^* - (\bar{\partial}^t c_{j\bar{k}}(\omega))^*.$$

Notice that $\alpha^* = *\bar{\alpha}*$ if α is an one-form, by a direct computation, we have

$$(3.5) \quad \begin{aligned} ([d^E, \delta_{[V_j, \bar{V}_k]}]u, v) &= (c_{j\bar{k}}(\omega)u, \bar{\partial}^t(\bar{\partial}^t)^*v) - (c_{j\bar{k}}(\omega)u, \partial^{E_t}(\partial^{E_t})^*v) \\ &\quad + (c_{j\bar{k}}(\omega)(\partial^{E_t})^*u, (\partial^{E_t})^*v) - (c_{j\bar{k}}(\omega)(\bar{\partial}^t)^*u, (\bar{\partial}^t)^*v) \\ &\quad + (c_{j\bar{k}}(\omega)(\partial^{E_t})^*\partial^{E_t}u, v) - (c_{j\bar{k}}(\omega)(\bar{\partial}^t)^*\bar{\partial}^t u, v) \\ &\quad + (c_{j\bar{k}}(\omega)\bar{\partial}^t u, \bar{\partial}^t v) - (c_{j\bar{k}}(\omega)\partial^{E_t}u, \partial^{E_t}v). \end{aligned}$$

Thus if u is an (n, q) -form and $c_{j\bar{k}}(\omega) \equiv 0$ then we have

$$(3.6) \quad (\Theta_{j\bar{k}}u, v) = (\Theta(E, h)(V_j, \bar{V}_k)u, v) + (\bar{\partial}V_j|_{X_t \lrcorner} u, \bar{\partial}V_k|_{X_t \lrcorner} v).$$

Now let us **control the second fundamental form** $(P^\perp D_{t^j}u, P^\perp D_{t^k}v)$. Since each $P^\perp D_j u$ is the L^2 -minimal solution of

$$(3.7) \quad \bar{\partial}^t(\cdot) = \bar{\partial}^t D_{t^j}u,$$

similar as the product case, one may also use the Hörmander's L^2 -theory to control the norm of $P^\perp D_j u$. By definition, we have

$$(3.8) \quad D_{t^j} \mathbf{u} = [\partial^E, \delta_{V_j}] \mathbf{u}.$$

Since

$$(3.9) \quad [\bar{\partial}, [\partial^E, \delta_{V_j}]] + [\partial^E, [\bar{\partial}, \delta_{V_j}]] + [\delta_{V_j}, [\bar{\partial}, \partial^E]] = 0,$$

we have

$$(3.10) \quad \bar{\partial}^t D_{t^j}u = -[\partial^{E_t}, \bar{\partial}V_j|_{X_t}]u - (V_j \lrcorner \Theta(E, h))|_{X_t} \wedge u.$$

If each $\partial^{E_t}u_j = 0$ then we know that $a := -\sum P^\perp D_j u_j$ is the L^2 -minimal solution of

$$(3.11) \quad \bar{\partial}^t(\cdot) = \partial^{E_t}b + c,$$

where

$$(3.12) \quad b := \sum \bar{\partial}V_j|_{X_t \lrcorner} u_j, \quad c := \sum (V_j \lrcorner \Theta(E, h))|_{X_t} \wedge u_j.$$

By Hörmander's L^2 -theory, $i\Theta(E_t, h^t) \wedge (\omega^t)^q \geq 0$ implies that

$$(3.13) \quad \|a\|^2 \leq \|b\|^2 + \lim_{\varepsilon \rightarrow 0} (Q_\varepsilon^{-1}c, c), \quad Q_\varepsilon := [i\Theta(E_t, h^t), \Lambda_{\omega^t}] + \varepsilon.$$

Thus we have

$$(3.14) \quad \sum (\Theta_{j\bar{k}}^\mathcal{K} u_j, u_k) \geq \sum (\Theta(E, h)(V_j, \bar{V}_k)u_j, u_k) - \lim_{\varepsilon \rightarrow 0} (Q_\varepsilon^{-1}c, c).$$

By Lemma 3.10 in [34], we know that the right hand side is non-negative. Similar proof works for other parts of this theorem, please see [34] for the details. \square

By a similar argument, one may also prove the following:

Theorem 3.2 (Yamaguchi's theorem for a proper Kähler fibration). *Assume that ω is Kähler, $c_{j\bar{k}}(\omega) \equiv 0$ and $i\Theta(E, h) \wedge \omega^q \geq 0$. Assume further that the dimension of $H^{n,q}(E_t)$ is a constant. If g is a holomorphic section of the quotient bundle $\mathcal{I}^{n,q+1}$, and*

$$(3.15) \quad (D_{t^j}g)(t) = ([\partial^E, \delta_{V_j}]\mathbf{g})|_{X_t} \equiv 0.$$

Then $-||g||_G^2$ is a smooth plurisubharmonic function on B .

4. TWISTED VERSION OF GRIFFITHS' THEOREM

We will give a short account of a recent joint work with Berndtsson and Paun. We shall show how to look at it by using the relative $\bar{\partial}$ -complex. Let $\pi : \mathcal{X} \rightarrow B$ be a proper fibration, let E be a holomorphic vector bundle over the total space \mathcal{X} . By Theorem 2.3 in [13] or [34], if $\dim H^{p,q}(E_t)$ does not depend on t then one may look at

$$\mathcal{H}^{p,q} := R^q \pi_* \mathcal{O}(\wedge^p T_{\mathcal{X}/B}^* \otimes E),$$

as the holomorphic quotient bundle $\mathcal{K}^{p,q}/\mathcal{I}^{p,q}$. Notice that the quotient norm of a class $[u^t]$ in $\mathcal{K}_t^{p,q}/\mathcal{I}_t^{p,q}$ is equal to

$$(4.1) \quad ||[u^t]|| := \inf\{||u^t + v^t|| : v^t \in \mathcal{I}_t^{p,q}\}.$$

By the Hodge theory, we know that

$$(4.2) \quad \inf\{||u^t + v^t|| : v^t \in \mathcal{I}_t^{p,q}\} = ||\mathbb{H}u^t||,$$

where $\mathbb{H}u^t$ denotes the $\bar{\partial}^t$ -harmonic part of u^t . We shall also write

$$(4.3) \quad \mathbb{H}^\perp u^t := u^t - \mathbb{H}u^t.$$

Let us denote by $\Theta_{j\bar{k}}^{\mathcal{H}}$ the curvature operators on $\mathcal{H}^{p,q}$. By the curvature formula for the quotient bundle, we have:

Theorem 4.1. *Assume that the total space \mathcal{X} is Kähler. Assume further that $\dim H^{p,q}(E_t)$ does not depend on t . Then we have*

$$(4.4) \quad (\Theta_{j\bar{k}}^{\mathcal{H}}[u], [v]) = (\Theta_{j\bar{k}}^{\mathcal{K}} \mathbb{H}u, \mathbb{H}v) + \left(\mathbb{H}^\perp (\bar{\partial}_{t\bar{k}} \mathbb{H}u), \mathbb{H}^\perp (\bar{\partial}_{t\bar{j}} \mathbb{H}v) \right),$$

where $[u], [v]$ are smooth sections of $\mathcal{H}^{p,q}$.

Now let us consider the following special cases:

- **A:** ω is Kähler and $\Theta(E, h) \equiv 0$;
- **B:** $p + q = n$, E is a line bundle and $i\Theta(E, h) = \pm\omega$.

By the Hodge theory, we know that in both cases, the space of $\bar{\partial}^t$ -harmonic E_t -valued (p, q) -forms is equal to the space of ∂^{E_t} -harmonic E_t -valued (p, q) -forms. Thus

$$(4.5) \quad (d^E \mathbb{H}u)|_{X_t} \equiv 0,$$

which implies that

$$(4.6) \quad ([d^E, \delta_{[V_j, \bar{V}_k]}] \mathbb{H}u, \mathbb{H}v) \equiv 0.$$

Thus by (3.3), we have

$$(4.7) \quad ([L_j, L_{\bar{k}}] \mathbb{H}u, \mathbb{H}v) \equiv (\Theta(E, h)(V_j, \bar{V}_k)u, v).$$

Assume further that B is one dimensional. Put

$$(4.8) \quad \Theta_{t\bar{t}}^{\mathcal{H}} := \Theta_{1\bar{1}}^{\mathcal{H}}, \quad \bar{\partial}_t := \bar{\partial}_{t\bar{1}}, \quad D_t := D_{t\bar{1}}, \quad V := V_1, \quad A := (\Theta(E, h)(V, \bar{V})u, u).$$

By (3.2), (3.1) and Theorem 4.1, we have

$$(4.9) \quad (\Theta_{t\bar{t}}^{\mathcal{H}}[u], [u]) = A + ||\bar{\partial}V|_{X_t \lrcorner} \mathbb{H}u||^2 - ||\partial\bar{V}|_{X_t \lrcorner} \mathbb{H}u||^2 + ||\mathbb{H}^\perp(\bar{\partial}_t \mathbb{H}u)||^2 - ||P^\perp(D_t \mathbb{H}u)||^2$$

We shall use the following proposition in [13]:

Proposition 4.2. *Assume that **A** or **B** is true. Then $P^\perp(D_t \mathbb{H}u)$ is the L^2 -minimal solution of $\bar{\partial}^t(\cdot) = -\partial^{E_t}(\bar{\partial}V|_{X_t \lrcorner} \mathbb{H}u)$ and $\mathbb{H}^\perp(\bar{\partial}_t \mathbb{H}u)$ is the L^2 -minimal solution of $\partial^{E_t}(\cdot) = -\bar{\partial}^t(\partial\bar{V}|_{X_t \lrcorner} \mathbb{H}u)$.*

Let us denote by \square the $\bar{\partial}^t$ -Laplace and denote by $\bar{\square}$ the ∂^{E_t} -Laplace. By the Bochner-Kodaira-Nakano formula, in case **A**, we have $\square\alpha = \bar{\square}\alpha$, in case **B**, we have $\square\alpha = \bar{\square}\alpha \pm \alpha$ if $i\Theta(E, h) = \pm\omega$. Thus (4.9), Proposition 4.2 and page 15 in [5] together imply the following twisted version of Griffiths' theorem in [13]:

Theorem 4.3 (Twisted version of Griffiths' theorem). *Assume that ω is Kähler and $\dim H^{p,q}(E_t)$ does not depend on t . Assume further that B is one dimensional. If $\Theta(E, h) \equiv 0$ then we have the following Griffiths formula:*

$$(4.10) \quad (\Theta_{t\bar{t}}^{\mathcal{H}}[u], [u]) = \|\mathbb{H}(\bar{\partial}V|_{X_t \lrcorner} \mathbb{H}u)\|^2 - \|\mathbb{H}(\partial\bar{V}|_{X_t \lrcorner} \mathbb{H}u)\|^2.$$

If $p + q = n$, E is a line bundle and $i\Theta(E, h) = \omega$ then

$$(4.11) \quad (\Theta_{t\bar{t}}^{\mathcal{H}}[u], [u]) \geq (|V|_{\omega}^2 \mathbb{H}u, \mathbb{H}u) + \|\mathbb{H}(\bar{\partial}V|_{X_t \lrcorner} \mathbb{H}u)\|^2 - \|\mathbb{H}(\partial\bar{V}|_{X_t \lrcorner} \mathbb{H}u)\|^2.$$

If $p + q = n$, E is a line bundle and $i\Theta(E, h) = -\omega$ then

$$(4.12) \quad (\Theta_{t\bar{t}}^{\mathcal{H}}[u], [u]) \leq -(|V|_{\omega}^2 \mathbb{H}u, \mathbb{H}u) + \|\mathbb{H}(\bar{\partial}V|_{X_t \lrcorner} \mathbb{H}u)\|^2 - \|\mathbb{H}(\partial\bar{V}|_{X_t \lrcorner} \mathbb{H}u)\|^2.$$

REFERENCES

- [1] B. Berndtsson, *Subharmonicity properties of the Bergman kernel and some other functions associated to pseudoconvex domains*, Ann. Inst. Fourier (Grenoble), **56** (2006), 1633–1662.
- [2] B. Berndtsson, *Curvature of vector bundles associated to holomorphic fibrations*, Ann. Math. **169** (2009), 531–560.
- [3] B. Berndtsson, *Positivity of direct image bundles and convexity on the space of Kähler metrics*, J. Diff. Geom. **81** (2009), 457–482.
- [4] B. Berndtsson, *An introduction to things $\bar{\partial}$* , Analytic and algebraic geometry, IAS/Park City Math. Ser., vol. 17, Amer. Math. Soc., Providence, RI, 2010, 7–76.
- [5] B. Berndtsson, *Strict and nonstrict positivity of direct image bundles*, Math. Z. **269** (2011), 1201–1218.
- [6] B. Berndtsson, *Convexity on the space of Kähler metrics*. Ann. Fac. Sci. Toulouse Math. **22** (2013), 713–746.
- [7] B. Berndtsson, *The openness conjecture for plurisubharmonic functions*, arxiv: 1305.5781.
- [8] B. Berndtsson, *The Openness Conjecture and Complex Brunn-Minkowski Inequalities*, Complex Geometry and Dynamics, Volume 10 of the series Abel Symposia, 2013, 29–44.
- [9] B. Berndtsson, *A Brunn-Minkowski type inequality for Fano manifolds and some uniqueness theorems in Kähler geometry*, Invent. Math. **200** (2014), 149–200.
- [10] B. Berndtsson and R. J. Berman, *Convexity of the K-energy on the space of Kähler metrics and uniqueness of extremal metrics*. arXiv:1405.0401 [math.CV].
- [11] B. Berndtsson and L. Lempert, *A proof of the Ohsawa-Takegoshi theorem with sharp estimates*, arXiv:1407.4946 [math.CV].
- [12] B. Berndtsson and M. Paun, *Bergman kernel and the pseudoeffectivity of relative canonical bundles*, Duke Math. J. **145** (2008), 341–378.
- [13] B. Berndtsson, M. Paun and X. Wang, *Iterated Kodaira-Spencer map and its applications*, preprint.
- [14] Z. Blocki, *Suita conjecture and the Ohsawa-Takegoshi extension theorem*, Invent. Math. **193** (2013), 149–158.
- [15] B. Y. Chen, *A simple proof of the Ohsawa-Takegoshi extension theorem*, arXiv:1105.2430.
- [16] T. Geiger, G. Schumacher, *Curvature of higher direct image sheaves*, arXiv:1501.07070v1 [math.AG]
- [17] Q. A. Guan and X. Y. Zhou, *A solution of an L^2 extension problem with an optimal estimate and applications*, Ann. Math. **181** (2015), 1139–1208.
- [18] R. S. Hamilton, *Deformation of complex structures on manifolds with boundary. I. The stable case*, J. Diff. Geom. **12** (1977), 1–45.
- [19] R. S. Hamilton, *Deformation of complex structures on manifolds with boundary. II. Families of noncoercive boundary value problems*, J. Diff. Geom. **14** (1979), 409–473.
- [20] L. Hörmander, *L^2 -estimates and existence theorems for the $\bar{\partial}$ -operator*, Acta Math. **113** (1965), 89–152.
- [21] L. Hörmander, *An introduction to complex analysis in several variables*. Vol. 7. Elsevier, 1973.
- [22] K. T. Kim, N. Levenberg and H. Yamaguchi, *Robin functions for complex manifolds and applications*. American Mathematical Soc., 2011.

- [23] L. Lempert and R. Szöke, *Direct Images, Fields of Hilbert Spaces, and Geometric Quantization*, Communications in Mathematical Physics **327** (2014), 49–99.
- [24] K. Liu and X. Yang, *Curvatures of direct image sheaves of vector bundles and applications*, J. Diff. Geom. **98** (2014), 117–145.
- [25] F. Maitani, *Variations of meromorphic differentials under quasiconformal deformations*, J. Math. Kyoto Univ. **24** (1984), 49–66.
- [26] F. Maitani, H. Yamaguchi, *Variation of Bergman metrics on Riemann surfaces*, Math. Ann. **330** (2004), 477–489.
- [27] C. Mourougane, S. Takayama, *Hodge metrics and positivity of direct images*, J. Reine Angew. Math. **606** (2007), 167–178.
- [28] C. Mourougane, S. Takayama, *Hodge metrics and the curvature of higher direct images*, Ann. Sci. Éc. Norm. Supér. **41** (2008), 905–924.
- [29] T. Ohsawa, K. Takegoshi, *On the extension of L^2 -holomorphic functions*, Math. Z. **195** (1987), 197–204.
- [30] G. Schumacher, *Positivity of relative canonical bundles and applications*, Invent. Math. **190** (2012), 1–56.
- [31] H. Tsuji, *Variation of Bergman kernels of adjoint line bundles*, arXiv:0511342 [math.CV].
- [32] X. Wang, *A curvature formula associated to a family of pseudoconvex domains*, arXiv:1508.00242, to appear in Annales de l’Institut Fourier.
- [33] X. Wang, *Subharmoncity properties of the L^2 -minimal solution of the $\bar{\partial}$ -equation*, preprint.
- [34] X. Wang, *Curvature of higher direct image sheaves and its application on negative-curvature criterion for the Weil-Petersson metric*, preprint.
- [35] H. Yamaguchi, *Variations of pseudoconvex domains over \mathbb{C}^n* , The Michigan Mathematical Journal, **36** (1989), 415–457.

SCHOOL OF MATHEMATICAL SCIENCES, FUDAN UNIVERSITY, SHANGHAI, 200433, CHINA

CURRENT ADDRESS: DEPARTMENT OF MATHEMATICAL SCIENCES, CHALMERS UNIVERSITY OF TECHNOLOGY AND UNIVERSITY OF GOTHENBURG. SE-412 96 GOTHENBURG, SWEDEN

E-mail address: wangxu1113@gmail.com

E-mail address: xuwa@chalmers.se